

# ON THE ASYMPTOTIC PLATEAU'S PROBLEM FOR CMC HYPERSURFACES ON RANK 1 SYMMETRIC SPACES OF NONCOMPACT TYPE

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**ABSTRACT.** Let  $M^n$ ,  $n \geq 3$ , be a Hadamard manifold with strictly negative sectional curvature  $K_M \leq -\alpha$ ,  $\alpha > 0$ . Assume that  $M$  satisfies the *strict convexity condition* at infinity according to [18] (see also the definition below) and, additionally, that  $M$  admits a *helicoidal* one parameter subgroup  $\{\varphi_t\}$  of isometries (i.e. there exists a geodesic  $\gamma$  of  $M$  such that  $\varphi_t(\gamma(s)) = \gamma(t+s)$  for all  $s, t \in \mathbb{R}$ ). We then prove that, given a compact topological  $\{\varphi_t\}$ -shaped hypersurface  $\Gamma$  in the asymptotic boundary  $\partial_\infty M$  of  $M$  (that is, the orbits of the extended action of  $\{\varphi_t\}$  to  $\partial_\infty M$  intersect  $\Gamma$  at one and only one point), and given  $H \in \mathbb{R}$ ,  $|H| < \sqrt{\alpha}$ , there exists a complete properly embedded constant mean curvature (CMC)  $H$  hypersurface  $S$  of  $M$  such that  $\partial_\infty S = \Gamma$ .

This result extends Theorem 1.8 of B. Guan and J. Spruck [11] to more general ambient spaces, as rank 1 symmetric spaces of noncompact type, and allows  $\Gamma$  to be  $\{\varphi_t\}$ -shaped with respect to more general one parameter subgroup of isometries  $\{\varphi_t\}$  of the ambient space. For example, in  $\mathbb{H}^n$ ,  $\Gamma$  can be *loxodromic* -shaped, where loxodromic is a curve in  $\mathbb{S}^{n-1} = \partial_\infty \mathbb{H}^n$  that makes a constant angle with a family of circles connecting two points of  $\mathbb{S}^{n-1}$ . A fundamental result used to prove our main theorem, which has interest on its own, is the extension of the interior gradient estimates for CMC Killing graphs proved in Theorem 1 of [7] to CMC graphs of Killing submersions.

## 1. INTRODUCTION

Let  $M^n$  be a Cartan-Hadamard manifold (namely a simply connected, complete Riemannian manifold with nonpositive sectional curvature) of dimension  $n \geq 3$ .

The asymptotic boundary  $\partial_\infty M$  of  $M$  is defined as the set of all equivalence classes of unit speed geodesic rays in  $M$ ; two such rays  $\gamma_1, \gamma_2 : [0, \infty) \rightarrow M$  are equivalent if  $\sup_{t \geq 0} d(\gamma_1(t), \gamma_2(t)) < \infty$ , where

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The first author was supported by the CNPq (Brazil) project 501559/2012-4. The second author was supported by the CNPq (Brazil) project 302955/2011-9.

$d$  is the Riemannian distance in  $M$ . The so called *geometric* compactification  $\overline{M}$  of  $M$  is then given by  $\overline{M} := M \cup \partial_\infty M$ , endowed with the cone topology (see [9] or [19], Ch. 2). For any subset  $S \subset M$ , we define  $\partial_\infty S = \partial_\infty M \cap \overline{S}$ .

The asymptotic Plateau problem for  $k$  ( $\geq 2$ ) dimensional area minimizing submanifolds in  $M$  consists in finding, for a given a  $k - 1$  dimensional, closed, topological submanifold  $\Gamma$  of  $\partial_\infty M$ , a locally area minimizing, complete submanifold  $S^k$  of  $M$  such that  $\partial_\infty S = \Gamma$ .

By using methods from the Geometric Measure Theory, this problem was first studied in the hyperbolic space by M.T. Anderson [2] and his results extended to Gromov hyperbolic manifolds by U. Lang and V. Bangert ([3], [13], [14]).

Within the framework of the classical Plateau problem, the second author of the present paper with F. Tomi [20] study the asymptotic problem for minimal disk type surfaces in a general Hadamard manifold  $M$ .

In codimension 1, given  $H \in \mathbb{R}$ , we may consider the asymptotic Plateau's problem for the constant mean curvature (CMC)  $H$  hypersurface in  $M$ , namely, given a compact topological hypersurface  $\Gamma \subset \partial_\infty M$ , find a complete CMC  $H$  hypersurface  $S$  of  $M$  ( $H$ -hypersurface, for short) such that  $\partial_\infty S = \Gamma$ . This problem has also attracted the attention of many mathematicians more recently. The results of M.T. Anderson [2] have been extended to the CMC case by Y. Tonegawa [21] and H. Alencar and H. Rosenberg in [1].

Both Geometric Measure theory and Plateau's technique are methods that lead, in general, to the existence of hypersurfaces with singularities. Thus, a natural question, raised by B. Guan and J. Spruck in [11], asks about the existence of a smooth constant mean curvature hypersurface asymptotic to  $\Gamma$  at infinity in  $\mathbb{H}^n$ . This problem in fact had already been studied earlier in the minimal case by F. H. Lin [15].

A way to obtain smooth solutions is by finding a suitable system of coordinates in order to write the hypersurface as a graph, and then to use standard elliptic PDE methods. In [15], F.H. Lin represented the hypersurfaces in the half space model of  $\mathbb{H}^n$  as vertical graphs, that is, in the usual way of  $\mathbb{R}_+^n$  when using the cartesian system of coordinates.

The results of F.H. Lin [15] were extended to the CMC case by B. Nelli and J. Spruck in [16] where they proved the existence of a smooth CMC  $|H| < 1$  hypersurface in the hyperbolic space  $\mathbb{H}^n$  with sectional curvature  $-1$  if  $\Gamma$  is assumed to be convex and compact. Later, also using PDE's techniques, B. Guan and J. Spruck [11] (see also [8] for a different approach based on a variational method) improved

the convexity condition by requiring a starshaped property of  $\Gamma$ . We refer the reader to the nice survey of B. Coskunuzer [5], where the references of many other closely related papers to this subject can be found.

In both papers [16] and [11] the authors used the underlying Euclidean structure of the half space model for  $\mathbb{H}^n$  to state the convexity and starshaped properties of  $\Gamma$ . However, although the convexity is not an intrinsic notion of the hyperbolic geometry, the starshapeness of  $\Gamma$  is. It can be formulated in intrinsic terms using the conformal structure of  $\overline{\mathbb{H}}^n$  by requiring  $\Gamma$  to be “circle shaped”, meaning that there are two points  $p_1, p_2 \in \mathbb{S}^{n-1} = \partial_\infty \mathbb{H}^n$  such that any arc of circle from  $p_1$  to  $p_2$  intersects  $\Gamma$  at one and only one point. A limit circle shaped condition, where  $p_1 = p_2$ , was also introduced and used by the second author in [17] to ensure the existence of a smooth CMC hypersurface having  $\Gamma$  as asymptotic boundary (see the Introduction and Theorem 6 of [17] for a detailed description of this case).

In the present paper we extend Theorem 1.8 of [11] in two directions. First, we allow  $\Gamma$  to be “shaped” with respect to a more general one parameter subgroup of conformal diffeomorphisms of  $\mathbb{S}^{n-1} = \partial_\infty \mathbb{H}^n$ . Secondly, we allow the ambient space to be any rank 1 symmetric space of noncompact type. Both results are consequences of a more general theorem that holds in a Hadamard manifold endowed with some special Killing field.

As we shall see in the proof ahead, the Killing field allows to introduce a special system of coordinates which is quite suitable for using standard elliptic PDE techniques. To write down precise statements we first introduce some general notions and terminology.

Let  $\gamma : (-\infty, \infty) \rightarrow M$  be an arc length geodesic. We say that a one parameter subgroup of isometries  $\{\varphi_t^\gamma\}$  of  $M$  associated to  $\gamma$  is *helical* if  $\varphi_t^\gamma(\gamma(s)) = \gamma(t+s)$  for all  $s, t \in \mathbb{R}$ . In the sequel, since there is no possibility of confusion, we shall omit the dependance of  $\{\varphi_t^\gamma\}$  with respect to the geodesic  $\gamma$ .

Let us illustrate the previous definition with a simple case that justifies this terminology: If  $M = \mathbb{R}^3$  then any helical one parameter subgroup of isometries, up to a conjugation, is of the form

$$\varphi_t(x, y, z) = \left( \begin{bmatrix} \cos at & \sin at \\ -\sin at & \cos at \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, z + t \right)$$

for some  $a \in \mathbb{R}$ . When  $a = 0$ ,  $\{\varphi_t\}$  is a one parameter subgroup of transvections along the  $z$ -axis. More generally, a one parameter subgroup of transvections along a geodesic in a symmetric space (see [12]) is a particular case of helical one parameter subgroup of isometries.

Since the equivalence relation between geodesics and convergent sequences are preserved under isometries, the action of  $\{\varphi_t\}$  on  $M$  extends to the compactification  $\overline{M}$  of  $M$  and the extended action is continuous. The orbits of  $\{\varphi_t\}$  are the curves  $O(x) := \{\varphi_t(x) \mid t \in \mathbb{R}\}$  where  $x \in \overline{M}$ . Observe that  $\{\varphi_t\}$  has two singular orbits in  $\overline{M}$ , namely,  $O(\gamma(\pm\infty))$ , where  $\gamma$  is the geodesic translated by  $\{\varphi_t\}$ .

Finally, we will also need to use the *Strictly Convexity Condition* (“SC condition”) introduced in [20]. We say that  $M$  satisfies the SC condition if, given  $x \in \partial_\infty M$  and a relatively open subset  $W \subset \partial_\infty M$  containing  $x$ , there exists a  $C^2$  open set  $\Omega \subset \overline{M}$  such that  $x \in \text{Int}(\partial_\infty \Omega) \subset W$  and  $M \setminus \Omega$  is convex, where  $\text{Int}(\partial_\infty \Omega)$  stands for the interior of  $\partial_\infty \Omega$  in  $\partial_\infty M$ .

We are now in position to state our main result :

**Theorem 1.** *Let  $M$  be a Hadamard manifold with sectional curvature  $K_M \leq -\alpha$ , for some  $\alpha > 0$ , satisfying the SC condition. Let  $\{\varphi_t\}$  be a helicoidal one parameter subgroup of isometries of  $M$ . Let  $\Gamma \subset \partial_\infty M$  be a compact topological embedded  $\{\varphi_t\}$ -shaped hypersurface of  $\partial_\infty M$ , that is, any nonsingular orbit of  $\{\varphi_t\}$  in  $\partial_\infty M$  intersects  $\Gamma$  at one and only one point. Then, given  $H \in \mathbb{R}$ ,  $|H| < \sqrt{\alpha}$ , there exists a complete, properly embedded  $H$ -hypersurface  $S$  of  $M$  such that  $\partial_\infty S = \Gamma$ . Moreover any orbit of  $\{\varphi_t\}$  intersects  $S$  at one and only one point.*

We point out that the SC condition is satisfied by a large class of manifolds. For example, if the sectional curvature is bounded from above by a strictly negative constant and decreases at most exponentially (see Theorem 14 of [18]) then the SC condition is satisfied. In particular, it is satisfied by any rank 1 symmetric spaces of noncompact type. Therefore, as an immediate consequence of the previous theorem, we obtain:

**Corollary 2.** *Assume that  $M$  is a rank 1 symmetric space of noncompact type and assume that the sectional curvature of  $M$  is bounded by  $-\alpha$ ,  $\alpha > 0$ . Let  $\{\varphi_t\}$  be a one parameter of transvections of  $M$ . Let  $\Gamma \subset \partial_\infty M$  be a compact embedded topological  $\{\varphi_t\}$ -shaped hypersurface of  $\partial_\infty M$ . Then, given  $H \in \mathbb{R}$ ,  $|H| < \sqrt{\alpha}$ , there exists a complete, properly embedded  $H$ -hypersurface  $S$  of  $M$  such that  $\partial_\infty S = \Gamma$ . Moreover any orbit of  $\{\varphi_t\}$  intersects  $S$  at one and only one point.*

Finally we point out an interesting corollary of Theorem 1 in the case where  $M = \mathbb{H}^n$ , the hyperbolic space of constant sectional curvature  $-1$ . Recalling that a *loxodromic curve* is a curve in  $\mathbb{S}^{n-1}$  that intersects with a constant angle  $\theta$  any arc of circle of  $\mathbb{S}^{n-1}$  connecting two fixed points of  $\mathbb{S}^{n-1}$  (see [22]). These curves are induced by one-parameter

subgroups of isometries of  $\mathbb{H}^n$  of helidoidal type. For example, in the half space model  $z > 0$  of  $\mathbb{H}^3$ , up to conjugation, they are of the form

$$\varphi_t(x, y, z) = e^t \left( \begin{bmatrix} \cos \theta t & \sin \theta t \\ -\sin \theta t & \cos \theta t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, z \right).$$

**Corollary 3.** *Let  $0 \leq \theta < \pi/2$  and  $p_1, p_2 \in \mathbb{S}^{n-1} = \partial_\infty \mathbb{H}^n$  be two distinct points of  $\mathbb{S}^{n-1}$ . Let  $L_\theta$  be the family of loxodromic curves that intersects any arc of circle from  $p_1$  to  $p_2$  with a constant angle  $\theta$ . Let  $\Gamma \subset \mathbb{S}^{n-1}$  be a compact embedded topological  $L_\theta$ -shaped hypersurface of  $\mathbb{S}^{n-1}$ . Then, given  $H \in \mathbb{R}$ ,  $|H| < 1$ , there exists a complete, properly embedded  $H$ -hypersurface  $S$  of  $\mathbb{H}^n$  such that  $\partial_\infty S = \Gamma$ .*

We notice that, taking  $\theta = 0$  in the previous corollary, we recover Theorem 1.8 of [11]. Theorem 1.8 also follows from Corollary 2 since radial graphs (considered in [11]) are transvections along a geodesic of  $\mathbb{H}^n$ .

A fundamental result for proving the above theorems, which has interest on its own, are the interior gradient estimates of the solutions of the CMC  $H$  graph PDE for Killing submersions (see Theorem 4 below). It extends Theorem 1 of [7].

## 2. PROOFS OF THE RESULTS

In what follows we use most of the nomenclature and the results proved by M. Dajczer and J. H. de Lira in [6]. However, we introduce the notion of a Killing graph on a manner slightly different from the one considered in [6].

For the next result we allow  $M$  be any Riemannian manifold and  $Y$  a Killing field in  $M$  without singularities. Denote by  $\mathcal{O}(x)$  the integral curve (which we also call orbit) of  $Y$  through a point  $x \in M$ . By a complete  $Y$ -Killing section (we shall refer only to a Killing section because  $Y$  will be fixed throughout the text) we mean a complete up to the boundary (possibly empty) hypersurface  $P$  of  $M$  such that any orbit  $\mathcal{O}(p)$  of  $Y$  through a point  $p$  of  $P$  intersects  $P$  only at  $p$  and the intersection is transversal. We call  $\Omega := P \setminus \partial P$  a Killing domain. If  $P = \overline{\Omega}$  is a hypersurface of class  $C^{2,\alpha}$  in  $M$  then we say that  $\Omega$  is a  $C^{2,\alpha}$  Killing domain.

If  $u$  is a function defined on a subset  $T$  of  $P$ , the Killing graph of  $u$  is given by

$$\text{Gr}(u) = \{\varphi(u(p), p) \mid p \in T\}$$

where  $\varphi(s, x) = \varphi_s(x)$  is the flow of  $Y$ . In the sequel,  $s$  will stand for the flow parameter. We also set

$$\Gamma_T = \{\varphi(s, x) \mid x \in T \text{ and } s \in \mathbb{R}\}.$$

Next, we denote by  $\Pi : M \rightarrow P$  the projection defined by  $\Pi(x) = \mathcal{O}(x) \cap P$ . In all the sequel, we endow  $P$  with the Riemannian metric  $\langle \cdot, \cdot \rangle_\Pi$  such that  $\Pi$  becomes a Riemannian submersion.

Assume that  $\Omega$  is a  $C^{2,\alpha}$  Killing domain. Given  $H \in \mathbb{R}$ , it is not difficult to show that  $\text{Gr}(u)$  has CMC  $H$  with respect to the unit normal vector field  $\eta$  to  $\text{Gr}(u)$  such that  $\langle Y, \eta \rangle \leq 0$  if and only if  $u$  satisfies a certain second order quasi-linear elliptic PDE  $Q_H[u] = 0$  on  $M$  in terms of the metric  $\langle \cdot, \cdot \rangle_\Pi$  in  $P$  (for details, including an explicit expression of  $Q_H$ , see Section 2.1 of [6] or the short revision done below).

We may then refer to the CMC  $H$  Dirichlet problem in a Killing domain  $\Omega \subset M$  and for a given boundary data  $\phi \in C^0(\partial\Omega)$  as the PDE boundary problem

$$(1) \quad \begin{cases} Q_H[u] = 0 \text{ in } \Omega & u \in C^{2,\alpha}(\Omega) \cap C^0(\overline{\Omega}) \\ u|_{\partial\Omega} = \phi. \end{cases}$$

We begin by obtaining interior gradient estimates for the solutions of (1). Our result generalizes Theorem 1 of [7] to the case of CMC  $H$  graph PDE of Killing submersions.

Fix a point  $o \in \Omega$  and let  $r > 0$  be such that  $r < i(o)$ , the injectivity radius of  $M$  at  $o$ . We obtain the following result :

**Theorem 4.** *Let  $\Omega$  be a Killing domain in  $M$ . Let  $o \in \Omega$  and  $r > 0$  such that the open geodesic ball  $B_r(o)$  is contained in  $\Omega$ . Let  $u \in C^3(B_r(o))$  be a negative solution of  $Q[u] = 0$  in  $B_r(o)$ . Then there is a constant  $L$  depending only on  $u(o)$ ,  $r$ ,  $|Y|$  and  $H$  such that  $|\nabla u(o)| \leq L$ .*

Before proving the above theorem, we review the nomenclature and some facts of [6].

We fix a local reference frame  $v_1, \dots, v_n$  on  $\overline{\Omega}$  and we set  $\sigma_{ij} = \langle v_i, v_j \rangle_\Pi$ . We will now define a local frame in  $M$ . We denote by  $D_1, \dots, D_n$  the basic vector fields  $\Pi$ -related to  $v_1, \dots, v_n$ . The frame  $D_0, \dots, D_n$ , we considered on  $M$ , is defined by  $D_0 = f^{\frac{1}{2}} \partial_s$ , where  $f = \frac{1}{|Y|^2}$ ,  $(\partial_s(q) = \varphi_*(s, p) \partial_s(p))$ , and  $D_i(q) = \varphi_*(s, p) D_i(p)$ , where  $q = \varphi(s, p)$ ,  $p \in P$ . We point out that the unit normal vector field to  $\text{Gr}(u)$  pointing upward is given by

$$(2) \quad N = \frac{1}{W} (f^{\frac{1}{2}} D_0 - \hat{u}^j D_j),$$

where  $\hat{u}^j = \sigma^{ij} D_i(u - s)$  and  $W^2 = f + \hat{u}^i \hat{u}_i = f + \sigma_{ij} \hat{u}^i \hat{u}^j$ . We notice that  $\hat{u}_i$  and  $W$  are not depending on  $s$  and therefore can be seen as function defined on  $P$ . Finally, using the previous notation,

the operator  $Q$  (defined in (1)) can be written as

$$Q_H[u] = \frac{1}{W}(A^{ij}\hat{u}_{j;i} - \frac{(f+W^2)}{W^2}\langle \Pi_*\bar{\nabla}_{D_0}D_0, Du \rangle) - nH,$$

where  $\hat{u}_{i;j} = \langle \bar{\nabla}_{D_i}\bar{\nabla}(u-s), D_j \rangle$ ,  $Du = \Pi_*\bar{\nabla}(u-s)$  and  $A^{ik} = \sigma^{ik} - \frac{\hat{u}^i\hat{u}^k}{W^2}$ .

*Proof of Theorem 4.* The proof will follow closely the one of Theorem 1 in [7]. Let  $p \in B_r(o)$  be an interior point where  $h = \eta W$  attains its maximum, where  $\eta$  is a smooth function with support in  $B_r(o)$  which will be determined in the sequel. In all this section, the computations will be done at the point  $p$ . Let  $v_1, \dots, v_n$  be an orthonormal tangent frame at  $p \in B_r(o)$ . Then we have  $h_i = 0$  (where the derivative is taken with respect to  $v_i$ ). This implies that

$$(3) \quad \eta_i W = -\eta W_i.$$

We also have, since  $\frac{A^{ij}}{W}$  is definite positive, that

$$0 \geq \frac{1}{W}A^{ij}h_{ij} = \frac{1}{W}A^{ij}(W\eta_{i;j} + 2\eta_i W_j + \eta W_{i;j}).$$

Using (3), the previous inequality can be rewritten as

$$(4) \quad A^{ij}\eta_{i;j} + \frac{\eta}{W^2}A^{ij}(WW_{i;j} - 2W_i W_j) \leq 0.$$

From (2), we have

$$(5) \quad N^k = -\frac{\hat{u}^k}{W}.$$

Derivating  $W$ , we find

$$W_i = \frac{f_i}{2W} + \frac{\hat{u}^k \hat{u}_{k;i}}{W} = \frac{f_i}{2W} - N^k \hat{u}_{k;i}.$$

From (5), we get

$$N_{i;j}^k = -\frac{\hat{u}_{i;j}^k}{W} + \frac{\hat{u}^k W_j}{W^2}.$$

Using the previous inequalities, we have

$$\begin{aligned}
W_{i;j} &= \frac{f_{i;j}}{2W} - \frac{f_i W_j}{2W^2} - N_{;j}^k \hat{u}_{k;i} - N^k \hat{u}_{k;ij} \\
&= \frac{f_{i;j}}{2W} + \frac{\hat{u}_{;j}^k \hat{u}_{k;i}}{W} - \frac{\hat{u}^k \hat{u}_{k;i} W_j}{W^2} - \frac{f_i W_j}{2W^2} - N^k \hat{u}_{k;ij} \\
&= \frac{f_{i;j}}{2W} + \frac{\hat{u}_{;j}^k \hat{u}_{k;i}}{W} - \frac{W_i W_j}{W} - N^k \hat{u}_{k;ij} \\
&= \frac{f_{i;j}}{2W} + \frac{A^{kl}}{W} \hat{u}_{l;j} \hat{u}_{k;i} + \frac{f_i f_j}{4W^3} - \frac{1}{2W^2} (W_i f_j + W_j f_i) - N^k \hat{u}_{k;ij}.
\end{aligned}$$

Multiplying by  $A^{ij}$  the above equation and using (3), we find

$$(6) \quad A^{ij} W_{i;j} = \frac{A^{ij} f_{i;j}}{2W} + \frac{A^{ij} A^{kl}}{W} \hat{u}_{l;j} \hat{u}_{k;i} + \frac{A^{ij} f_i f_j}{4W^3} + \frac{1}{\eta W} A^{ij} \eta_i f_j - A^{ij} N^k \hat{u}_{k;ij}.$$

In order to get rid of the term involving three derivatives of  $u$  in (6), we want to find a commutation formula for  $\hat{u}_{k;ij}$ . We recall (see equation (11) of [6]) that

$$\hat{u}_{k;i} = u_{k;i} - s_{k;i} + \frac{1}{2} \gamma_{ki},$$

where  $\gamma_{ki} = f^{\frac{1}{2}} \langle [D_k, D_i], D_0 \rangle$ . We deduce from the previous equality that

$$\begin{aligned}
\hat{u}_{k;ij} &= u_{k;ij} - s_{k;ij} + \frac{1}{2} (\gamma_{ki})_j = u_{i;jk} + R_{kji}^l u_l - s_{k;ij} + \frac{1}{2} (\gamma_{ki})_j \\
&= (\hat{u}_{j;i} + s_{j;i} - \frac{1}{2} \gamma_{ji})_k + R_{kji}^l u_l - s_{k;ij} + \frac{1}{2} (\gamma_{ki})_j \\
&= \hat{u}_{j;ik} + s_{j;ik} - \frac{1}{2} (\gamma_{ji})_k + R_{kji}^l u_l - s_{k;ij} + \frac{1}{2} (\gamma_{ki})_j \\
&= \hat{u}_{j;ik} + R_{kji}^l \hat{u}_l + R_{kji}^l s_l + s_{j;ik} - s_{k;ij} + \frac{1}{2} ((\gamma_{ki})_j - (\gamma_{ji})_k) \\
&= \hat{u}_{j;ik} + R_{kji}^l \hat{u}_l + C_{ijk},
\end{aligned}$$

where  $C_{ijk} = R_{kji}^l s_l + s_{j;ik} - s_{k;ij} + \frac{1}{2} ((\gamma_{ki})_j - (\gamma_{ji})_k)$  is not depending on  $u$ . Using (1) and the commutation formula, the last term of (6)



rewrites as

$$\begin{aligned}
A^{ij} N^k \hat{u}_{k;ij} &= A^{ij} N^k \hat{u}_{j;ik} - \frac{A^{ij} R_{kji}^l \hat{u}_l \hat{u}^k}{W} - \frac{\hat{u}^k A^{ij} C_{ijk}}{W} \\
&= N^k (A^{ij} \hat{u}_{j;i})_k - N^k A_{,k}^{ij} \hat{u}_{i;j} - \frac{A^{ij} R_{kji}^l \hat{u}_l \hat{u}^k}{W} - \frac{\hat{u}^k A^{ij} C_{ijk}}{W} \\
&= n N^k (WH)_k + N^k \left( \frac{(f + W^2)}{W^2} \langle \Pi_* \bar{\nabla}_{D_0} D_0, Du \rangle \right)_k \\
&\quad - N^k A_{,k}^{ij} \hat{u}_{j;i} - \frac{A^{ij} R_{kji}^l \hat{u}_l \hat{u}^k}{W} - \frac{\hat{u}^k A^{ij} C_{ijk}}{W}.
\end{aligned}$$

Straightforward computations using (3) give

$$(WH)_k = W_k H + W H_k = \frac{W}{\eta} (-\eta_k H + \eta H_k),$$

and

$$(7) \quad \left( \frac{f + W^2}{W^2} \right)_k = \frac{f_k}{W^2} - \frac{f}{W^4} (f_k + 2 \hat{u}^l \hat{u}_{l;k}) = \frac{1}{W^2} (f_k + 2 \frac{f \eta_k}{\eta}).$$

We also have, using (7),

$$\begin{aligned}
\left( \frac{(f + W^2)}{W^2} \langle \Pi_* \bar{\nabla}_{D_0} D_0, Du \rangle \right)_k &= \frac{(f + W^2)}{2fW^2} \left[ \left( \frac{f_l f_k}{f} - f_{k;l} \right) W N^l + f^l \hat{u}_{l;k} \right] \\
&\quad + \langle \Pi_* \bar{\nabla}_{D_0} D_0, Du \rangle \frac{1}{W^2} (f_k + 2 \frac{f \eta_k}{\eta}),
\end{aligned}$$

and

$$\begin{aligned}
A_{,k}^{ij} &= -\frac{1}{W^2} (\hat{u}_{,k}^i \hat{u}^j + \hat{u}^i \hat{u}_{,k}^j) + \frac{1}{W^4} (f_k - 2W N^l \hat{u}_{l;k}) \hat{u}^i \hat{u}^j \\
&= \frac{1}{W} (\hat{u}_{,k}^i - N^i N^l \hat{u}_{l;k}) N^j + \frac{1}{W} (\hat{u}_{,k}^j - N^j N^l \hat{u}_{l;k}) N^i + \frac{1}{W^2} f_k N^i N^j \\
&= \frac{1}{W} A^{il} \hat{u}_{l;k} N^j + \frac{1}{W} A^{jl} \hat{u}_{l;k} N^i + \frac{1}{W^2} f_k N^i N^j.
\end{aligned}$$

Multiplying the previous equality by  $N^k \hat{u}_{j;i}$ , we find

$$N^k A_{,k}^{ij} \hat{u}_{j;i} = \frac{1}{W} N^k \hat{u}_{j;i} \hat{u}_{l;k} (A^{il} N^j + A^{jl} N^i) + \frac{1}{W^2} f_k N^i N^j N^k \hat{u}_{j;i}.$$

Recalling that

$$N^k \hat{u}_{k;i} = \frac{f_i}{2W} + \frac{W \eta_i}{\eta},$$

and

$$\hat{u}_{i;j} = \hat{u}_{j;i} + \gamma_{ij},$$

we have

$$\begin{aligned}
N^k A_{;k}^{ij} \hat{u}_{j;i} &= \frac{1}{W^2} f_k N^k N^i \left( \frac{f_i}{2W} + \frac{W\eta_i}{\eta} \right) + \frac{1}{W} A^{il} \left( \frac{f_i}{2W} + \frac{W\eta_i}{\eta} \right) N^k (\hat{u}_{k;l} + \gamma_{lk}) \\
&\quad + \frac{1}{W} A^{jl} N^k N^i (\hat{u}_{i;j} + \gamma_{ji}) (\hat{u}_{k;l} + \gamma_{lk}) \\
&= \frac{1}{W^2} f_k N^k N^i \left( \frac{f_i}{2W} + \frac{W\eta_i}{\eta} \right) + \frac{2}{W} A^{il} \left( \frac{f_i}{2W} + \frac{W\eta_i}{\eta} \right) \left( \frac{f_l}{2W} + \frac{W\eta_l}{\eta} \right) \\
&\quad + \frac{1}{W} A^{jl} N^k N^i \gamma_{ji} \gamma_{lk} + \frac{3}{W} A^{il} \left( \frac{f_i}{2W} + \frac{W\eta_i}{\eta} \right) N^k \gamma_{lk},
\end{aligned}$$

and

$$N^k f_l \hat{u}_{l;k} = N^k f_l (\hat{u}_{k;l} - \gamma_{kl}) = \frac{f}{2W} \sigma^{kl} \frac{f_k f_l}{f} + \frac{W}{\eta} f_l \eta^l - \gamma_{kl} N^k f_l.$$

Using the previous computations, we deduce that the last term of (6) can be rewritten as

$$\begin{aligned}
A^{ij} N^k \hat{u}_{k;ij} &= n N^k \frac{W}{\eta} (-\eta_k H + \eta H_k) - \frac{2}{W} A^{il} \left( \frac{f_i}{2W} + \frac{W\eta_i}{\eta} \right) \left( \frac{f_l}{2W} + \frac{W\eta_l}{\eta} \right) \\
&\quad - \frac{1}{W^2} f_k N^k N^i \left( \frac{f_i}{2W} + \frac{W\eta_i}{\eta} \right) - \frac{1}{W} A^{jl} N^k N^i \gamma_{ji} \gamma_{lk} \\
&\quad - \frac{3}{W} A^{il} \left( \frac{f_i}{2W} + \frac{W\eta_i}{\eta} \right) N^k \gamma_{lk} - \frac{A^{ij} R_{kji}^l \hat{u}_l \hat{u}^k}{W} \\
&\quad - \frac{\hat{u}^k A^{ij} C_{ijk}}{W} + \langle \Pi_* \bar{\nabla}_{D_0} D_0, Du \rangle \frac{N_k}{W^2} (f_k + 2 \frac{f \eta_k}{\eta}) \\
&\quad + \frac{(f + W^2)}{2fW^2} \left[ \left( \frac{f}{2W} \sigma^{kl} + W N^k N^l \right) \frac{f_k f_l}{f} - W N^k N^l f_{k;l} + \frac{W}{\eta} f_l \eta^l - \gamma_{kl} N^k f_l \right].
\end{aligned}$$

Thus, from (6), we obtain

$$\begin{aligned}
A^{ij} W_{ij} - \frac{2}{W} A^{ij} W_i W_j &= \frac{3}{4W^3} A^{ij} f_i f_j + \frac{1}{W} A^{ij} A^{kl} \hat{u}_{l;j} \hat{u}_{k;i} + \frac{3}{W\eta} A^{ij} f_i \eta_j + \frac{1}{2W} A^{ij} f_{i;j} \\
&\quad - n N^k \frac{W}{\eta} (-\eta_k H + \eta H_k) + \frac{1}{W^2} f_k N^k N^i \left( \frac{f_i}{2W} + \frac{W\eta_i}{\eta} \right) + \frac{1}{W} A^{jl} N^k N^i \gamma_{ji} \gamma_{lk} \\
&\quad + \frac{3}{W} A^{il} \left( \frac{f_i}{2W} + \frac{W\eta_i}{\eta} \right) N^k \gamma_{lk} + \frac{A^{ij} R_{kji}^l \hat{u}_l \hat{u}^k}{W} \\
&\quad + \frac{\hat{u}^k A^{ij} C_{ijk}}{W} - \langle \Pi_* \bar{\nabla}_{D_0} D_0, Du \rangle \frac{N_k}{W^2} (f_k + 2 \frac{f \eta_k}{\eta}) \\
&\quad - \frac{(f + W^2)}{W^2} \frac{1}{2f} \left[ \left( \frac{f}{2W} \sigma^{kl} + W N^k N^l \right) \frac{f_k f_l}{f} - W N^k N^l f_{k;l} + \frac{W}{\eta} f_l \eta^l - \gamma_{kl} N^k f_l \right].
\end{aligned}$$

Multiplying by  $\frac{\eta}{W}$ , we have

$$\begin{aligned}
& \frac{\eta}{W}(A^{ij}W_{ij} - \frac{2}{W}A^{ij}W_iW_j) \\
& \geq \left[ -nN^kH_k - \frac{f_k}{W^3}N^k \langle \Pi_* \bar{\nabla}_{D_0} D_0, Du \rangle + \frac{1}{2W^2}A^{ij}f_{i;j} \right. \\
& + \frac{1}{W^2}A^{jl}N^kN^i\gamma_{ji}\gamma_{lk} + \frac{3}{2W^3}A^{il}f_iN^k\gamma_{lk} + \frac{\hat{u}^kA^{ij}C_{ijk}}{W^2} + \frac{A^{ij}R_{kji}^l\hat{u}_l\hat{u}^k}{W^2} \\
& - \frac{(f+W^2)}{W^2}\frac{1}{2f} \left[ \left( \frac{f}{2W^2}\sigma^{kl} + N^kN^l \right) \frac{f_kf_l}{f} - N^kN^lf_{k;l} - \frac{1}{W}\gamma_{kl}N^kf_l \right] \Big] \eta \\
& + \left[ \left( nH + \frac{1}{W^2}N^kf_k - \frac{2f}{W^3} \langle \Pi_* \bar{\nabla}_{D_0} D_0, Du \rangle \right) N^i \right. \\
& \quad \left. + \frac{3}{W}A^{jl}N^k\gamma_{lk} + \left( \frac{3}{W^2}A^{ij} - \frac{(f+W^2)}{W^2}\frac{1}{2f}\sigma^{ij} \right) f_j \right] \eta_i.
\end{aligned}$$

Thus it is easy to see that there exists a constant  $M > 0$ , not depending on  $u$ , such that

$$\frac{\eta}{W^2}(WA^{ij}W_{ij} - 2A^{ij}W_iW_j) \geq -M\eta - A^i\eta_i,$$

where  $A^i$  is the coefficient of  $\eta_i$ . From (4), we deduce that

$$(8) \quad A^{ij}\eta_{i;j} - M\eta - A^i\eta_i \leq 0.$$

We are now ready to choose an explicit  $\eta$ . We take

$$\eta(x) = g(\phi(x)) = e^{C_1\phi(x)} - 1 = e^{C_1(1 - \frac{d^2(x)}{r^2} + \frac{u(x)}{C})^+} - 1,$$

where  $C = -\frac{1}{2u(o)}$ . Straightforward computations give

$$\eta_i = g'(-r^{-2}(d^2)_i + C(u_i - s_i)) = g'(-r^{-2}(d^2)_i + C\hat{u}_i),$$

and

$$\eta_{i;j} = g'(-r^{-2}(d^2)_{i;j} + C\hat{u}_{i;j}) + g''(-r^{-2}(d^2)_i + C\hat{u}_i)(-r^{-2}(d^2)_j + C\hat{u}_j).$$

We deduce from the two previous lines that

$$A^{ij}(-r^{-2}(d^2)_i + C\hat{u}_i)(-r^{-2}(d^2)_j + C\hat{u}_j) \geq \frac{C^2f}{W^2} \left( |Du|^2 - \frac{2}{Cr^2} \langle Du, \nabla d^2 \rangle \right),$$

and

$$\begin{aligned}
A^{ij}(-r^{-2}(d^2)_{i;j} + C\hat{u}_{i;j}) &= -r^{-2}A^{ij}(d^2)_{i;j} \\
&+ C \left( nWH + \frac{f+W^2}{W^2} \langle \Pi_* \bar{\nabla}_{D_0} D_0, Du \rangle + A^{ij}\gamma_{ij} \right),
\end{aligned}$$

where

$$A^{ij}(d^2)_{i;j} = \Delta(d^2) - \frac{1}{W^2} \langle \nabla_{Du} \nabla d^2, Du \rangle.$$

Inserting the previous expressions into (8), we have

$$\begin{aligned} & \frac{C^2 f}{W^2} \left( |Du|^2 - \frac{2}{Cr^2} \langle Du, \nabla d^2 \rangle \right) g'' \\ & + \left[ C \left( nWH + \frac{f + W^2}{W^2} \langle \Pi_* \bar{\nabla}_{D_0} D_0, Du \rangle + A^{ij} \gamma_{ij} \right) \right. \\ & \quad \left. - r^{-2} \left( \Delta(d^2) - \frac{1}{W^2} \langle \nabla_{Du} \nabla d^2, Du \rangle \right) \right] g' \\ & \leq Mg + A^i (-r^2 (d^2)_i + C \hat{u}_i) g'. \end{aligned}$$

Using the explicit expression of  $A^i$ , it is easy to see that  $CA^i \hat{u}_i$  contains bounded terms and the term

$$C \left( nWH + \frac{f + W^2}{W^2} \langle \Pi_* \bar{\nabla}_{D_0} D_0, Du \rangle \right).$$

Therefore, we conclude that

$$\frac{C^2 f}{W^2} \left( |Du|^2 - \frac{2}{Cr^2} \langle Du, \nabla d^2 \rangle \right) g'' + Pg' - Mg \leq 0,$$

where  $P$  and  $M$  do not depend on  $u$ . Finally, it is easy to check that the coefficient of  $g''$  is strictly positive if we assume that  $|Du| \geq \frac{16u_0}{r}$ . It implies that

$$W(p) \leq C_2 = \sup_{B_r(o)} f + \frac{16u_0}{r}.$$

Since  $p$  is the maximum point of  $h$ , this implies that

$$(e^{\frac{C_1}{2}} - 1)W(0) \leq C_2 e^{C_1}.$$

□

For the proof of Theorem 1 we make use of the following lemma, which shows that the SC condition implies an explicit mean convexity condition. Precisely:

**Lemma 5.** *Assume  $M$  is a Hadamard manifold satisfying the strict convexity condition and such that  $K_M \leq -\alpha$ , for some constant  $\alpha > 0$ . Then  $M$  satisfies the  $h$ -mean convexity condition for  $h < \sqrt{\alpha}$ , that is, given  $x \in \partial_\infty M$ , a relatively open subset  $W \subset \partial_\infty M$  containing  $x$  and  $h < \sqrt{\alpha}$ , there exists a  $C^2$  open set  $\Lambda \subset \overline{M}$  such that  $x \in \text{Int}(\partial_\infty \Lambda) \subset W$  and the mean curvature of  $M \setminus \Lambda$  with respect to the normal vector pointing to  $M \setminus \Lambda$  is bigger than or equal to  $h$ .*

*Proof.* Given  $x \in \partial_\infty M$  and a relatively open subset  $W \subset \partial_\infty M$  containing  $x$ , let  $\Omega$  be a convex unbounded domain in  $M$ , given by the SC condition such that  $x \in \text{Int}(\partial_\infty \Omega) \subset W$ . Denote by  $d : \Omega \rightarrow \mathbb{R}$  the distance function to  $\partial\Omega$ . Then the hessian comparison theorem (see [4]) yields

$$\Delta d \geq (n-1)\sqrt{\alpha} \tanh(\sqrt{\alpha}d),$$

i.e. the equidistant hypersurface  $\Omega_d$  of  $\Omega$  is  $\sqrt{\alpha} \tanh(\sqrt{\alpha}d)$ -convex. Since  $\tanh(\sqrt{\alpha}d) \xrightarrow{d \rightarrow \infty} 1$ , we deduce that  $M$  also satisfies the  $h$ -mean convexity condition for  $h < \sqrt{\alpha}$ .  $\square$

*Proof of Theorem 1.* Let  $\gamma : (-\infty, \infty) \rightarrow M$  be the geodesic translated by  $Y$ . Set  $P = \exp_o \{Y(o)\}^\perp$  where  $o = \gamma(0)$ . Let  $p \in P$  and  $t \in \mathbb{R}$  be given. We may write  $p = \exp_{\gamma(s)} u$  for some  $s \in \mathbb{R}$  and  $u \in \gamma'(s)^\perp$ . Since  $\tilde{\gamma}(r) = \exp_{\gamma(s)}(ru)$ ,  $r \in [0, 1]$ , is a geodesic and  $\varphi_t$  an isometry,  $\beta(r) := \varphi_t(\tilde{\gamma}(r))$  is also a geodesic which, moreover, satisfies the initial conditions

$$\begin{aligned} \beta(0) &= \varphi_t(\tilde{\gamma}(0)) = \varphi_t(\gamma(s)) = \gamma(s+t) \\ \beta'(0) &= d(\varphi_t)_{\gamma(s)} u =: v, \end{aligned}$$

we have  $\beta(r) = \exp_{\gamma(s+t)}(rv)$  by uniqueness. It follows that

$$\varphi_t(p) = \beta(1) = \exp_{\gamma(s+t)} v.$$

Moreover, since

$$0 = \langle u, \gamma'(s) \rangle = \left\langle d(\varphi_t)_{\gamma(s)} u, d(\varphi_t)_{\gamma(s)} \gamma'(s) \right\rangle = \langle v, \gamma'(s+t) \rangle$$

we have  $v \in \gamma'(s+t)^\perp$  and, as the normal exponential map of a geodesic in Hadamard manifold is a diffeomorphism from the normal bundle of the geodesic onto  $M$ , we have  $\varphi_t(p) \cap P \neq \emptyset$  if and only if  $t = 0$ .

We now observe that  $Y$  is everywhere transversal to  $P$ . Indeed, assume by contradiction that  $Y$  is not transversal to  $P$  at some point  $p \in P$ . Let  $d : N \rightarrow \mathbb{R}$  be the distance function to  $P$ . We set  $f(t) = d(\varphi(t, p))$  and observe that  $f(0) = 0$ . Moreover, since  $\varphi(t, \cdot)$  is an isometry of  $N$ , we have, for any fixed  $t$ ,

$$\text{grad } d(\varphi(t, p)) = d\varphi(t, p)_p (\text{grad } d(p)).$$

Therefore, we obtain

$$\begin{aligned} f'(t) &= \left\langle \operatorname{grad} d, \frac{\partial \varphi(s, p)}{\partial s} \Big|_{s=t} \right\rangle = \langle \operatorname{grad} d(\varphi(t, p)), Y(\varphi(t, p)) \rangle \\ &= \left\langle d\varphi(t, p)_p(\operatorname{grad} d(p)), d\varphi(t, p)_p(Y(p)) \right\rangle \\ &= \langle \operatorname{grad} d(p), Y(p) \rangle = 0. \end{aligned}$$

This implies that  $f \equiv 0$  and, in return, that  $\varphi(t, p) \in P$  for all  $t$ , which yields to a contradiction. This proves that  $P$  is a Killing section.

Since any orbit of  $\varphi$  at  $\partial_\infty N$  intersects  $\Gamma$  at one and exactly one point,  $\Gamma$  is the Killing graph of a function  $\phi \in C^0(\partial_\infty P)$ . Let  $F \in C^{2,\alpha}(P) \cap C^0(\bar{P})$  ( $\bar{P} = P \cup \partial_\infty P$ ) be such that  $F|_{\partial_\infty P} = \phi$ .

Let  $\rho$  be the geodesic distance in  $P$  to a fixed point  $o \in P$ . We denote by  $B_k$ , for  $k = 2, 3, \dots$ , the geodesic ball in  $P$  centered in  $o$  and of radius  $k$ . We first show that, for any  $k = 2, 3, \dots$ , there is a solution  $u_k \in C^{2,\alpha}(\bar{B}_k)$  of

$$(9) \quad \begin{cases} Q_H[u_k] = 0, & \text{on } B_k \\ u_k|_{\partial B_k} = F_k = F|_{\partial B_k}. \end{cases}$$

In order to prove the existence of the  $u_k$ 's, we will need some a priori height estimate. More precisely, we claim that given some  $k \geq 2$ , there is a constant  $C_j$  depending only on  $j$  such that if  $u_k$  is a solution of (9) and  $j \leq k$  then  $\sup_{B_j} |u_k| \leq C_j$ . Let us prove the claim. We choose two open subsets  $U_\pm$  of  $\gamma(\pm\infty)$  in  $\partial_\infty M$ . Using the SC condition, we obtain the existence of two  $C^2$  convex subsets  $W_\pm$  of  $M$  such that  $\partial_\infty W_\pm \subset U_{K_\pm}$ . Denote by  $K_\pm$  the hypersurfaces  $K_\pm = \partial W_\pm$ . As observed in Lemma 5 and since  $|H| < \sqrt{\alpha}$ , we may assume that  $K_\pm$  are  $H_0$  mean convex with  $H_0 \geq H$ . We then choose  $C_j$  such that the orbit of  $\{\varphi_t\}$  through a point of  $B_j$  intersects  $W_\pm$  for some  $t \geq C_j$ . It is clear that we may assume that the Killing graph of  $F$  does not intersect  $K_\pm$ . The claim then follows from the tangency principle.

We have two important consequences of the previous height estimate. The first one is that problem (9) is solvable for any  $k \geq 2$ . In fact, the only missing hypothesis to apply Theorem 1 of [6] to guarantee the solvability of (9) are on the Ricci curvature of  $M$  and on the mean curvature of the Killing cylinder over the boundary of  $B_k$  (see [6], Theorem 1). Concerning the hypothesis on the mean curvature of the Killing cylinder over the boundary of  $B_k$ , we claim that it holds true in our setting. Indeed, since the orbits of  $\varphi_t$  are equidistant curves of  $\gamma$ , it follows that the Killing cylinder  $K_k$  over  $\partial B_k$  is an equidistant hypersurface of  $\gamma$ . Therefore the mean curvature  $H_{K_k}$  of  $K_k$  with respect to the inner normal vector field of  $K_k$  coincides with the Laplacian of the

distance to  $\gamma$ . One may then apply the hessian comparison theorem to obtain

$$H_{K_k} \geq \sqrt{\alpha} \tanh(k\sqrt{\alpha}) \geq H.$$

Moreover, a direct inspection on the proof of Theorem 1 of [6] shows that the hypothesis on the Ricci curvature is only used to obtain a priori height estimates, which we just obtained above.

Secondly, the a priori height estimates we obtained above, Theorem 4 and classical Schauder estimate for linear elliptic PDE (see [10]) guarantee the compactness of the sequence of solutions  $\{u_k\}$  on compact subsets of  $M$ . Then, by the diagonal method, the sequence  $\{u_k\}$  contains a subsequence converging uniformly in  $C^2$  norm on compact subsets of  $M$  to a global solution  $u \in C^\infty(P)$  of  $Q_H[u] = 0$ , where  $|H| < \sqrt{\alpha}$ . It remains to show that  $u$  extends continuously to  $\partial_\infty P$  and that  $u|_{\partial_\infty P} = \phi$ .

Let  $(x_k)_k$  be a sequence of points of  $P$  converging to  $x \in \partial_\infty P$ . Since  $\overline{P}$  is compact, there exists a subsequence  $\varphi(u(x_{k_j}), x_{k_j})$  of  $\varphi(u(x_k), x_k)$  which converges to  $z \in \overline{P}$ . Since  $x_k$  diverges and  $\varphi(u(x_{k_j}), x_{k_j}) \in Gr(u)$ , we have that  $z \in \partial_\infty Gr(u)$ . We claim that  $z \in Gr(\phi)$ . To prove the claim, we will show that if  $z \in \partial_\infty M \setminus Gr(\phi)$  then  $z \notin \partial_\infty Gr(u)$ . Let  $z \in \partial_\infty M \setminus Gr(\phi)$ . Since  $Gr(\phi)$  is compact and  $z \notin Gr(\phi)$ , using the SC condition, we can find an hypersurface  $E \subset M$  such that  $\partial_\infty E$  separates  $z$  and  $Gr(\phi)$ . Moreover, using Lemma 5, the mean curvature of  $E$  with respect to the unit normal vector field pointing to the connected component  $U$  of  $M \setminus E$  whose asymptotic boundary contains  $Gr(\phi)$ , is larger or equal to  $h$  for  $h < \sqrt{\alpha}$ . Since  $u_k|_{\partial B_k} \xrightarrow[k \rightarrow \infty]{} \phi$ , there exists  $k_0$  such that, for all  $k \geq k_0$ ,  $\partial Gr(u_k) \subset U$  and  $\partial Gr(u_k) \cap E = \emptyset$ . By the tangency principle and using that  $|H| < \sqrt{\alpha}$ , we deduce that, for all  $k \geq k_0$ ,  $Gr(u_k) \subset U$ . It follows that  $z \notin \partial_\infty Gr(u)$ . This proves the claim i.e.  $z \in Gr(\phi)$ . In particular, it follows that  $u$  is bounded.

Now, since  $\partial_\infty Gr(u) \subset Gr(\phi)$ , there exists  $x_0 \in \partial_\infty P$  such that  $z = \varphi(u(x_0), x_0)$ . Using that  $u$  is bounded, we deduce there exists a subsequence  $\{u(x_{k_{j_i}})\}$  which converges to some  $t_0 \in \mathbb{R}$ . It follows from the fact that the extension of  $\varphi_{t_0}$  to  $\overline{P}$  is continuous that

$$z = \lim_{i \rightarrow \infty} \varphi(u(x_{k_{j_i}}), x_{k_{j_i}}) = \varphi(t_0, x).$$

Since  $\varphi : \mathbb{R} \times \partial_\infty P \rightarrow \partial_\infty M$  is injective, we deduce that  $t_0 = \phi(x_0)$  and  $x_0 = x$ . Since this last fact holds true for every converging subsequences, we have proved that  $u(x_k) \xrightarrow[k \rightarrow \infty]{} \phi(x)$ . This concludes the proof of Theorem 1.  $\square$

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